

On the regularity criterion of weak solution for the 3D viscous Magneto-hydrodynamics equations

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Abstract. We improve and extend some known regularity criterion of weak solution for the 3D viscous Magneto-hydrodynamics equations by means of the Fourier localization technique and Bony's para-product decomposition.

Key words. MHD equations, Weak solution, Regularity criterion, Fourier Localization, Bony's para-product decomposition.

AMS subject classifications. 76W05 35B65

1 Introduction

In this paper, we consider the 3D incompressible magneto-hydrodynamics(MHD) equations

$$(MHD) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u = -\nabla p - \frac{1}{2} \nabla b^2 + b \cdot \nabla b, \\ \frac{\partial b}{\partial t} - \eta \Delta b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(0, x) = u_0(x), \quad b(0, x) = b_0(x). \end{cases} \quad (1.1)$$

Here u , b describe the flow velocity vector and the magnetic field vector respectively, p is a scalar pressure, $\nu > 0$ is the kinematic viscosity, $\eta > 0$ is the magnetic diffusivity, while u_0 and b_0 are the given initial velocity and initial magnetic field with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If $\nu = \eta = 0$, (1.1) is called the ideal MHD equations.

As same as the 3D Navier-Stokes equations, the regularity of weak solution for the 3D MHD equations remains open[17]. For the 3D Navier-Stokes equations, the Serrin-type criterion states that a Leray-Hopf weak solution u is regular provided the following condition holds[1, 9, 11, 16, 19]:

$$u \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \text{for} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 \leq p \leq \infty, \quad (1.2)$$

or

$$\nabla u \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \text{for } \frac{2}{q} + \frac{3}{p} \leq 2, \quad \frac{3}{2} < p \leq \infty. \quad (1.3)$$

Recently, Chen and Zhang [3] have refined the above conditions as follows: If there exists a small ε_0 such that for any $t \in (0, T)$, u satisfies

$$\limsup_{\varepsilon \rightarrow 0} \sup_j 2^{jsq} \int_{t-\varepsilon}^t \|\Delta_j u(\tau)\|_p^q d\tau \leq \varepsilon_0, \quad (1.4)$$

with $\frac{2}{q} + \frac{3}{p} = 1 + s$, $\frac{3}{1+s} < p \leq \infty$, $-1 < s \leq 1$, and $(p, s) \neq (\infty, 1)$, then u is regular in $(0, T] \times \mathbb{R}^3$, where Δ_j denotes the frequency localization operator. For the marginal case ($p = 3, q = \infty$), Cheskidov and Shvydkoy [6] have refined (1.2) to

$$u \in C([0, T]; B_{\infty, \infty}^{-1}). \quad (1.5)$$

Here $B_{\infty, \infty}^{-1}$ stands for the inhomogenous Besov spaces, see Section 2 for the definitions.

Wu [20, 21] extended some Serrin-type criteria for the Navier-Stokes equations to the MHD equations imposing conditions on both the velocity field u and the magnetic field b . However, some numerical experiments [15] seem to indicate that the velocity field plays the more important role than the magnetic field in the regularity theory of solutions to the MHD equations. Recently, He, Xin[12], and Zhou[24] have proved some regularity criteria to the MHD equations which do not impose any condition on the magnetic field b . Precisely, they showed that the weak solution remains smooth on $(0, T] \times \mathbb{R}^3$ if the velocity u satisfies one of the following conditions

$$u \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty; \quad (1.6)$$

$$u \in C([0, T]; L^3(\mathbb{R}^3)); \quad (1.7)$$

$$\nabla u \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} \leq 2, \quad \frac{3}{2} < p \leq \infty. \quad (1.8)$$

Meanwhile, inspired by the pioneering work of Constantin and Fefferman [7] where the regularity condition of the direction of vorticity was used to describe the regularity criterion to the Navier-Stokes equations, He and Xin [12] showed that the weak solution remains smooth on $(0, T] \times \mathbb{R}^3$ if the vorticity of the velocity $w = \nabla \times u$ satisfies the following condition

$$|w(x + y, t) - w(x, t)| \leq K|w(x + y, t)||y|^{\frac{1}{2}} \quad \text{if } |y| \leq \rho \quad |w(x + y, t)| \geq \Omega, \quad (1.9)$$

for $t \in [0, T]$ and three positive constants K, ρ, Ω .

For the marginal case $p = \infty$ in (1.8), Chen, Miao and Zhang [4] proved a Beale-Kato-Majda criterion in terms of the vorticity of the velocity u only by means of the Littlewood-Paley decomposition.

For the generalized MHD equations with fractional dissipative effect, Wu[22, 23] established some regularity results in terms of the velocity only.

The purpose of this paper is to improve and extend some known regularity criterion of weak solution for the MHD equations by means of the Fourier localization technique and Bony's para-product decomposition [2, 5]. Let us firstly recall the definition of weak solution.

Definition 1.1. The vector-valued function (u, b) is called a weak solution of (1.1) on $(0, T) \times \mathbb{R}^3$ if it satisfies the following conditions:

- (1) $(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;
- (2) $\operatorname{div} u = \operatorname{div} b = 0$ in the sense of distribution;
- (3) For any function $\psi(t, x) \in C_0^\infty((0, T) \times \mathbb{R}^3)$ with $\operatorname{div} \psi = 0$, there hold

$$\int_0^T \int_{\mathbb{R}^3} \{u \cdot \psi_t - \nu \nabla u \cdot \nabla \psi + \nabla \psi : (u \otimes u - b \otimes b)\} dx dt = 0,$$

and

$$\int_0^T \int_{\mathbb{R}^3} \{b \cdot \psi_t - \eta \nabla b \cdot \nabla \psi + \nabla \psi : (u \otimes b - b \otimes u)\} dx dt = 0.$$

Similar to the Navier-Stokes equation, the global existence of weak solutions to the MHD equations can be proved by using the Galerkin's method and compact argument, see[8]. Now we state our main result as follows.

Theorem 1.1. Let $(u_0, b_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, b) is a weak solution to (1.1) on $(0, T) \times \mathbb{R}^3$ with $0 \leq T \leq \infty$. If the velocity $u(t)$ satisfies

$$u(t) \in L^q(0, T; B_{p,\infty}^s), \quad (1.10)$$

with $\frac{2}{q} + \frac{3}{p} = 1 + s$, $\frac{3}{1+s} < p \leq \infty$, $-1 < s \leq 1$, and $(p, s) \neq (\infty, 1)$. Then the solution (u, b) is regular on $(0, T] \times \mathbb{R}^3$.

Remark 1.1. By the embedding $L^p \subsetneq B_{p,\infty}^0$, we see that our result is an improvement of (1.6) and (1.8). In addition, we establish the regularity criterion of weak solution for the MHD equation in the framework of Besov spaces with negative index in terms of the velocity only. On the other hand, the method in this paper can be applied to the generalized MHD equations, please refer to [22, 23] for details.

Remark 1.2. In the case of $s = 0$ or $s = 1$, Kozono, Ogawa and Taniuchi[14] proved the similar results for the Navier-Stokes equations by using the Logarithmic Sobolev inequality in the Besov spaces. However, if we try to use their method to our case, we can only obtain the regularity criterion in terms of both the velocity field u and the magnetic field b .

Remark 1.3. Chen, Miao and Zhang[4] proved the marginal case $(p, s) = (\infty, 1)$ by using a different argument. However, the method of [4] can't also be applied to the present case.

Remark 1.4. The regularity of weak solution (u, b) under the condition

$$u \in C(0, T; B_{\infty,\infty}^{-1}) \quad (1.11)$$

remains unknown. One easily checks that it is the special case of the endpoint case of (1.10) in Theorem 1.1 with $s = -1$.

Notation: Throughout the paper, C stands for a generic constant. We will use the notation $A \lesssim B$ to denote the relation $A \leq CB$ and the notation $A \approx B$ to denote the relations $A \lesssim B$ and $B \lesssim A$. Further, $\|\cdot\|_p$ denotes the norm of the Lebesgue space L^p .

2 Preliminaries

In this section, we are going to recall some basic facts on Littlewood-Paley theory, one may check [5] for more details.

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx.$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3.$$

Set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. Define the frequency localization operators:

$$\begin{aligned} \Delta_j f &= \varphi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x-y) dy, \quad \text{for } j \geq 0, \\ S_j f &= \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) f(x-y) dy, \quad \text{and} \\ \Delta_{-1} f &= S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2. \end{aligned} \tag{2.1}$$

Formally, $\Delta_j = S_{j+1} - S_j$ is a frequency projection into the annulus $\{|\xi| \approx 2^j\}$, and S_j is a frequency projection into the ball $\{|\xi| \lesssim 2^j\}$. One easily verifies that with the above choice of φ

$$\Delta_j \Delta_k f \equiv 0 \quad \text{if } |j-k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) \equiv 0 \quad \text{if } |j-k| \geq 5. \tag{2.2}$$

We now introduce the following definition of inhomogenous Besov spaces by means of Littlewood-Paley projection Δ_j and S_j :

Definition 2.1. Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, the inhomogenous Besov space $B_{p,q}^s$ is defined by

$$B_{p,q}^s = \{f \in \mathcal{S}'(\mathbb{R}^3); \quad \|f\|_{B_{p,q}^s} < \infty\},$$

where

$$\|f\|_{B_{p,q}^s} = \begin{cases} \left(\sum_{j=-1}^{\infty} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{\frac{1}{q}}, & \text{for } q < \infty, \\ \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L^p}, & \text{for } q = \infty. \end{cases}$$

Let us point out that $B_{2,2}^s$ is the usual Sobolev space H^s and that $B_{\infty,\infty}^s$ is the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$. We refer to [18] for more details.

We now recall the para-differential calculus which enables us to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see [2]). The paraproduct between u and v is defined by

$$T_u v \triangleq \sum_j S_{j-1} u \Delta_j v. \quad (2.3)$$

Formally, we have the following Bony's decomposition:

$$uv = T_u v + T_v u + R(u, v), \quad (2.4)$$

with

$$R(u, v) = \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v,$$

and we also denote

$$T'_u v \triangleq T_u v + R(u, v).$$

Let us conclude this section by recalling the Bernstein's inequality which will be frequently used in the proof of Theorem 1.1.

Lemma 2.1. [5] *Let $1 \leq p \leq q \leq \infty$. Assume that $f \in L^p$, then there exists a constant C independent of f, j such that*

$$\text{supp } \hat{f} \subset \{|\xi| \leq C2^j\} \implies \|\partial^\alpha f\|_{L^q} \leq C 3^{j|\alpha|+3j(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \quad (2.5)$$

$$\text{supp } \hat{f} \subset \left\{ \frac{1}{C} 2^j \leq |\xi| \leq C 2^j \right\} \implies \|f\|_{L^p} \leq C 2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_{L^p}. \quad (2.6)$$

3 Proof of Theorem 1.1

Since the weak solution $(u(t), b(t)) \in L^2(0, T; H^1(\mathbb{R}^3))$, for any time interval $(0, \delta)$, there exists an $\varepsilon \in (0, \delta)$ such that $(u(\varepsilon), b(\varepsilon)) \in H^1(\mathbb{R}^3)$. It is well known that there exist a maximal existence time $T_0 > 0$ and a unique strong solution

$$(\tilde{u}(t), \tilde{b}(t)) \in \mathcal{X}(\varepsilon, T_0) \triangleq C([\varepsilon, T_0]; H^1(\mathbb{R}^3)) \cap C^1((\varepsilon, T_0); H^1(\mathbb{R}^3)) \cap C((\varepsilon, T_0); H^3(\mathbb{R}^3))$$

which is the same as the weak solution (u, b) on (ε, T_0) [8, 17]. In order to complete the proof of Theorem 1.1, it suffices to show that the strong solution $(u(t), b(t))$ can be extended after $t = T_0$ in the class $\mathcal{X}(\varepsilon, T_0)$ under the condition of Theorem 1.1. For the convenience, we set $\nu = \eta = 1$ and $\varepsilon = 0$ in what follows. We denote

$$u_k = \Delta_k u, \quad b_k = \Delta_k b, \quad \pi_k = \Delta_k \pi,$$

here $\pi = p + \frac{1}{2}b^2$. We get by applying the operation Δ_k to both sides of (1.1) that

$$\begin{cases} \partial_t u_k - \Delta u_k + \Delta_k(u \cdot \nabla u) - \Delta_k(b \cdot \nabla b) = -\nabla \pi_k, \\ \partial_t b_k - \Delta b_k + \Delta_k(u \cdot \nabla b) - \Delta_k(b \cdot \nabla u) = 0. \end{cases} \quad (3.1)$$

Multiplying the first equation of (3.1) by u_k and the second one of (3.1) by b_k , we obtain by Lemma 2.1 for $k \geq 0$ that

$$\frac{1}{2} \frac{d}{dt} \|u_k(t)\|_2^2 + c2^{2k} \|u_k(t)\|_2^2 = -\langle \Delta_k(u \cdot \nabla u), u_k \rangle + \langle \Delta_k(b \cdot \nabla b), u_k \rangle, \quad (3.2)$$

$$\frac{1}{2} \frac{d}{dt} \|b_k(t)\|_2^2 + c2^{2k} \|b_k(t)\|_2^2 = -\langle \Delta_k(u \cdot \nabla b), b_k \rangle + \langle \Delta_k(b \cdot \nabla u), b_k \rangle. \quad (3.3)$$

Set

$$F_k(t) \triangleq (\|u_k(t)\|_2^2 + \|b_k(t)\|_2^2)^{\frac{1}{2}}.$$

Then we get by adding (3.2) and (3.3) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} F_k(t)^2 + c2^{2k} F_k(t)^2 &= -\langle \Delta_k(u \cdot \nabla u), u_k \rangle + \langle \Delta_k(b \cdot \nabla b), u_k \rangle \\ &\quad - \langle \Delta_k(u \cdot \nabla b), b_k \rangle + \langle \Delta_k(b \cdot \nabla u), b_k \rangle. \end{aligned} \quad (3.4)$$

Noting that

$$\begin{aligned} \langle u \cdot \nabla u_k, u_k \rangle &= \langle u \cdot \nabla b_k, b_k \rangle = 0, \\ \langle b \cdot \nabla b_k, u_k \rangle + \langle b \cdot \nabla u_k, b_k \rangle &= 0. \end{aligned}$$

The right hand side of (3.4) can be written as

$$\begin{aligned} &\langle [u, \Delta_k] \cdot \nabla u, u_k \rangle - \langle [b, \Delta_k] \cdot \nabla b, u_k \rangle + \langle [u, \Delta_k] \cdot \nabla b, b_k \rangle - \langle [b, \Delta_k] \cdot \nabla u, b_k \rangle \\ &\triangleq I + II + III + IV, \end{aligned}$$

where $[A, B] \triangleq AB - BA$. Using the Bony's decomposition (2.4), we rewrite I as

$$\begin{aligned} I &= \langle [T_{u^i}, \Delta_k] \partial_i u, u_k \rangle + \langle T'_{\Delta_k \partial_i u} u^i, u_k \rangle - \langle \Delta_k T_{\partial_i u} u^i, u_k \rangle - \langle \Delta_k R(u^i, \partial_i u), u_k \rangle \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

In view of the support of the Fourier transform of the term $T_{\partial_i u} u^i$, we have

$$\Delta_k T_{\partial_i u} u^i = \sum_{|k'-k| \leq 4} \Delta_k (S_{k'-1}(\partial_i u) u_{k'}^i).$$

This helps us to get by Lemma 2.1

$$|I_3| \lesssim \|u_k\|_2 \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} u\|_\infty \|u_{k'}\|_2. \quad (3.5)$$

Since $\operatorname{div} u = 0$, we have

$$\Delta_k R(u^i, \partial_i u) = \sum_{k', k'' \geq k-2; |k'-k''| \leq 1} \partial_i \Delta_k (\Delta_{k'} u^i \Delta_{k''} u).$$

This together with Lemma 2.1 yields

$$|I_4| \lesssim 2^k \|u_k\|_\infty \sum_{k' \geq k-2} \|u_{k'}\|_2^2. \quad (3.6)$$

Using the definition of $T'_{\Delta_k \partial_i u} u^i$, we have

$$T'_{\Delta_k \partial_i u} u^i = \sum_{k' \geq k-2} S_{k'+2} \Delta_k \partial_i u \Delta_{k'} u^i.$$

Note that $S_{k'+2} \Delta_k u = \Delta_k u$ for $k' > k$, we get

$$I_2 = \sum_{k-2 \leq k' \leq k} \langle S_{k'+2} \Delta_k \partial_i u \Delta_{k'} u^i, u_k \rangle,$$

from which and Lemma 2.1, it follows that

$$I_2 \lesssim \|u_k\|_2 \sum_{|k'-k| \leq 2} \|\nabla S_{k'-1} u\|_\infty \|u_{k'}\|_2 + 2^k \|u_k\|_\infty \sum_{k' \geq k-2} \|u_{k'}\|_2^2. \quad (3.7)$$

Making use of the definition of Δ_k , we have

$$\begin{aligned} [T_{u^i}, \Delta_k] \partial_i u &= \sum_{|k'-k| \leq 4} [S_{k'-1} u^i, \Delta_k] \partial_i u_{k'} \\ &= \sum_{|k'-k| \leq 4} 2^{3k} \int_{\mathbb{R}^3} h(2^k(x-y)) (S_{k'-1} u^i(x) - S_{k'-1} u^i(y)) \partial_i u_{k'}(y) dy \\ &= \sum_{|k'-k| \leq 4} 2^{4k} \int_{\mathbb{R}^3} \int_0^1 y \cdot \nabla S_{k'-1} u^i(x - \tau y) d\tau \partial_i h(2^k y) u_{k'}(x-y) dy, \end{aligned}$$

from which and the Minkowski inequality, we infer that

$$|I_1| \lesssim \|u_k\|_2 \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} u\|_\infty \|u_{k'}\|_2. \quad (3.8)$$

By summing up (3.5)-(3.8), we obtain

$$|I| \lesssim \|u_k\|_2 \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} u\|_\infty \|u_{k'}\|_2 + 2^k \|u_k\|_\infty \sum_{k' \geq k-2} \|u_{k'}\|_2^2. \quad (3.9)$$

Similar arguments as in deriving (3.9) can be used to get that

$$\begin{aligned} |II + IV| &\lesssim \|u_k\|_\infty \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} b\|_2 \|b_{k'}\|_2 + 2^k \|u_k\|_\infty \sum_{k' \geq k-2} \|b_{k'}\|_2^2 \\ &\quad + \|b_k\|_2 \sum_{|k'-k| \leq 4} (\|\nabla S_{k'-1} u\|_\infty \|b_{k'}\|_2 + \|\nabla S_{k'-1} b\|_2 \|u_{k'}\|_\infty) \\ &\quad + 2^k \|b_k\|_2 \sum_{k', k'' \geq k-2; |k'-k''| \leq 1} \|u_{k'}\|_\infty \|b_{k''}\|_2, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} |III| &\lesssim \|b_k\|_2 \sum_{|k'-k| \leq 4} (\|\nabla S_{k'-1} u\|_\infty \|b_{k'}\|_2 + \|\nabla S_{k'-1} b\|_2 \|u_{k'}\|_\infty) \\ &\quad + 2^k \|b_k\|_2 \sum_{k', k'' \geq k-2; |k'-k''| \leq 1} \|u_{k'}\|_\infty \|b_{k''}\|_2. \end{aligned} \quad (3.11)$$

Combining (3.9)-(3.11) with (3.4), we easily get for $k \geq 0$ that

$$\begin{aligned}
& \frac{d}{dt} F_k(t)^2 + 2^{2k} F_k(t)^2 \\
& \lesssim (\|u_k\|_2 + \|b_k\|_2) \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} u\|_\infty (\|u_{k'}\|_2 + \|b_{k'}\|_2) \\
& \quad + \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} b\|_2 (\|b_{k'}\|_2 \|u_k\|_\infty + \|u_{k'}\|_\infty \|b_k\|_2) \\
& \quad + 2^k \|b_k\|_2 \sum_{k', k'' \geq k-2; |k'-k''| \leq 1} \|u_{k'}\|_\infty \|b_{k''}\|_2 \\
& \quad + 2^k \|u_k\|_\infty \sum_{k' \geq k-2} (\|b_{k'}\|_2^2 + \|u_{k'}\|_2^2). \tag{3.12}
\end{aligned}$$

Making use of $B_{p,\infty}^{\frac{2}{q}+\frac{3}{p}-1}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{\frac{2}{q}-1}(\mathbb{R}^3)$, we only need to deal with the case when $p = +\infty$ since the other cases can be deduced from it by above Sobolev embedding. Here we omit the details. By the restrictions on p, q, s , we see that $s = \frac{2}{q} - 1$ and $q \in (1, +\infty)$.

Case 1. $q \in (1, 2]$.

Integrating (3.12) with respect to t , we deduce that

$$\begin{aligned}
& F_k(t)^2 - F_k(0)^2 + 2^{2k} \int_0^t F_k(\tau)^2 d\tau \\
& \lesssim \int_0^t (\|u_k\|_2 + \|b_k\|_2) \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} u\|_\infty (\|u_{k'}\|_2 + \|b_{k'}\|_2) d\tau \\
& \quad + \int_0^t \sum_{|k'-k| \leq 4} \|\nabla S_{k'-1} b\|_2 (\|b_{k'}\|_2 \|u_k\|_\infty + \|u_{k'}\|_\infty \|b_k\|_2) d\tau \\
& \quad + \int_0^t 2^k \|b_k\|_2 \sum_{k', k'' \geq k-2; |k'-k''| \leq 1} \|u_{k'}\|_\infty \|b_{k''}\|_2 d\tau \\
& \quad + \int_0^t 2^k \|u_k\|_\infty \sum_{k' \geq k-2} (\|b_{k'}\|_2^2 + \|u_{k'}\|_2^2) d\tau \\
& \triangleq \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4. \tag{3.13}
\end{aligned}$$

Take $\rho \in (\frac{1}{2}, 1)$ and set

$$A(t) \triangleq \sup_{k \geq -1} 2^{k\rho} F_k(\tau), \quad B(t) = \sup_{k \geq -1} 2^{2k(\rho+1)} \int_0^t F_k(\tau)^2 d\tau.$$

We get by using Lemma 2.1 that

$$\begin{aligned}
2^{2k\rho}\Pi_1 &\lesssim \int_0^t 2^{2k\rho}(\|u_k\|_2 + \|b_k\|_2) \sum_{|k'-k|\leq 4} \sum_{k''=-1}^{k'-2} \|u_{k''}\|_\infty 2^{k''}(\|u_{k'}\|_2 + \|b_{k'}\|_2) d\tau \\
&\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) 2^{2k\rho}(\|u_k\|_2 + \|b_k\|_2) \sum_{|k'-k|\leq 4} 2^{-k'\rho} \sum_{k''=-1}^{k'-2} 2^{k''(2-\frac{2}{q})} d\tau \\
&\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) 2^{k(\rho+2-\frac{2}{q})} F_k(\tau) d\tau \\
&\lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^2 d\tau \right)^{\frac{1}{q}} B(t)^{1-\frac{1}{q}}, \tag{3.14}
\end{aligned}$$

where we used the fact that $1 < q \leq 2$ in the last two inequalities. Similarly, we get by using Lemma 2.1 and the fact that $\rho < 1$ and $1 < q \leq 2$ that

$$\begin{aligned}
2^{2k\rho}\Pi_2 &\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} 2^{k(1-\frac{2}{q}+2\rho)} \sum_{|k'-k|\leq 4} \sum_{k''=-1}^{k'-2} \|b_{k''}\|_2 2^{k''}(\|b_{k'}\|_2 + \|b_k\|_2) d\tau \\
&\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) 2^{k(1-\frac{2}{q}+2\rho)} \sum_{|k'-k|\leq 4} 2^{k'(1-\rho)}(\|b_{k'}\|_2 + \|b_k\|_2) d\tau \\
&\lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^2 d\tau \right)^{\frac{1}{q}} B(t)^{1-\frac{1}{q}}, \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
2^{2k\rho}\Pi_3 &\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} 2^{k(2\rho+1)} \|b_k\|_2 \sum_{k'\geq k-2} \|b_{k'}\|_2 2^{k'(1-\frac{2}{q})} d\tau \\
&\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) 2^{k(2\rho+1)} \|b_k\|_2 \sum_{k'\geq k-2} 2^{k'(1-\frac{2}{q}-\rho)} d\tau \\
&\lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^2 d\tau \right)^{\frac{1}{q}} B(t)^{1-\frac{1}{q}}, \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
2^{2k\rho}\Pi_4 &\lesssim \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s} 2^{k(2\rho+2-\frac{2}{q})} \sum_{k'\geq k-2} (\|b_{k'}\|_2^2 + \|u_{k'}\|_2^2) d\tau \\
&\lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^2 d\tau \right)^{\frac{1}{q}} B(t)^{1-\frac{1}{q}}. \tag{3.17}
\end{aligned}$$

On the other hand, the strong solution (u, b) also satisfies the energy equality

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2 \int_0^t (\|\nabla u(\tau)\|_2^2 + \|\nabla b(\tau)\|_2^2) d\tau = \|u_0\|_2^2 + \|b_0\|_2^2,$$

hence, we have

$$\|\Delta_{-1}u(t)\|_2 + \|\Delta_{-1}b(t)\|_2 \leq C(\|u_0\|_2 + \|b_0\|_2). \tag{3.18}$$

Thus, summing up (3.13)-(3.18), we get by the Young's inequality that

$$A(t)^2 \leq C(A(0)^2 + \|u_0\|_2^2 + \|b_0\|_2^2) + C \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^2 d\tau,$$

which together with the Gronwall inequality yields that

$$A(t)^2 \leq C(A(0))^2 + \|u_0\|_2^2 + \|b_0\|_2^2 \exp\left(C \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q d\tau\right). \quad (3.19)$$

Case 2. $q \in (2, +\infty)$

Multiplying (3.12) by $2^{2k(q-1)}F_k(t)^{2(q-1)}$, then integrating the resulting equation with respect to t leads to the result for $k \geq 0$

$$\begin{aligned} & 2^{2k(q-1)}F_k(t)^{2q} - 2^{2k(q-1)}F_k(0)^{2q} + 2^{2kq} \int_0^t F_k(\tau)^{2q} d\tau \\ & \lesssim \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} (\|u_k\|_2 + \|b_k\|_2) \sum_{|k'-k|\leq 4} \|\nabla S_{k'-1}u\|_\infty (\|u_{k'}\|_2 + \|b_{k'}\|_2) d\tau \\ & \quad + \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} \sum_{|k'-k|\leq 4} \|\nabla S_{k'-1}b\|_2 (\|b_{k'}\|_2 \|u_k\|_\infty + \|u_{k'}\|_\infty \|b_k\|_2) d\tau \\ & \quad + \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} 2^k \|b_k\|_2 \sum_{k',k''\geq k-2; |k'-k''|\leq 1} \|u_{k'}\|_\infty \|b_{k''}\|_2 d\tau \\ & \quad + \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} 2^k \|u_k\|_\infty \sum_{k'\geq k-2} (\|b_{k'}\|_2^2 + \|u_{k'}\|_2^2) d\tau \\ & \triangleq \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4. \end{aligned} \quad (3.20)$$

Set

$$A(t) \triangleq \sup_{k \geq -1} 2^{(1-\frac{1}{q})k} F_k(t), \quad B(t) \triangleq \sup_{k \geq -1} 2^{2kq} \int_0^t F_k(\tau)^{2q} d\tau.$$

We obtain that by Lemma 2.1

$$\begin{aligned} \Pi_1 & \lesssim \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} (\|u_k\|_2 + \|b_k\|_2) \sum_{|k'-k|\leq 4} \sum_{k''=-1}^{k'-2} \|u_{k''}\|_\infty 2^{k''} (\|u_{k'}\|_2 + \|b_{k'}\|_2) d\tau \\ & \lesssim \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau)^2 d\tau \\ & \lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^{2q} d\tau \right)^{\frac{1}{q}} B(t)^{\frac{q-1}{q}}. \end{aligned} \quad (3.21)$$

Similarly, we get by using Lemma 2.1 and the fact that $q < +\infty$ that

$$\begin{aligned} \Pi_2 & \lesssim \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) 2^{k(1-\frac{2}{q})} \sum_{|k'-k|\leq 4} \sum_{k''=-1}^{k'-2} 2^{\frac{k''}{q}} (\|b_{k'}\|_2 + \|b_k\|_2) d\tau \\ & \lesssim \int_0^t 2^{2k(q-1)}F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) \sum_{|k'-k|\leq 4} 2^{k(1-\frac{1}{q})} (\|b_{k'}\|_2 + \|b_k\|_2) d\tau \\ & \lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^{2q} d\tau \right)^{\frac{1}{q}} B(t)^{\frac{q-1}{q}}, \end{aligned} \quad (3.22)$$

$$\begin{aligned}
\Pi_3 &\lesssim \int_0^t 2^{2k(q-1)} F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} \sum_{k' \geq k-2} \|b_{k'}\|_2 2^{k'(1-\frac{2}{q})} 2^k \|b_k\|_2 d\tau \\
&\lesssim \int_0^t 2^{2k(q-1)} F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau) \sum_{k' \geq k-2} 2^{-\frac{k'}{q}} 2^k \|b_k\|_2 d\tau \\
&\lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^{2q} d\tau \right)^{\frac{1}{q}} B(t)^{\frac{q-1}{q}}, \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
\Pi_4 &\lesssim \int_0^t 2^{2k(q-1)} F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} \\
&\quad \times \sum_{k' \geq k-2} (\|b_{k'}\|_2^2 + \|u_{k'}\|_2^2) 2^{k'(2-\frac{2}{q})} 2^{(k-k')(2-\frac{2}{q})} d\tau \\
&\lesssim \int_0^t 2^{2k(q-1)} F_k(\tau)^{2(q-1)} \|u(\tau)\|_{B_{\infty,\infty}^s} A(\tau)^2 d\tau \\
&\lesssim \left(\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^{2q} d\tau \right)^{\frac{1}{q}} B(t)^{\frac{q-1}{q}}. \tag{3.24}
\end{aligned}$$

Thus, combining (3.20)-(3.24) with (3.18) and using the Young's inequality lead to the result that

$$A(t)^{2q} \leq C(A(0)^{2q} + \|u_0\|_2^{2q} + \|b_0\|_2^{2q}) + C \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q A(\tau)^{2q} d\tau.$$

This together with the Gronwall inequality yields that

$$\sup_{t \in [0, T]} A(t)^{2q} \leq C(A(0)^{2q} + \|u_0\|_2^{2q} + \|b_0\|_2^{2q}) \exp \left(C \int_0^t \|u(\tau)\|_{B_{\infty,\infty}^s}^q d\tau \right). \tag{3.25}$$

By means of (3.19) and (3.25), it follows that there exists $\tilde{\rho} > \frac{1}{2}$ such that

$$\sup_{t \in [0, T_0]} (\|u(t)\|_{H^{\tilde{\rho}}} + \|b(t)\|_{H^{\tilde{\rho}}}) < +\infty,$$

by Sobolev embedding $B_{2,\infty}^\rho(\mathbb{R}^3) \hookrightarrow H^{\tilde{\rho}}(\mathbb{R}^3)$ with $\rho > \tilde{\rho}$ and $B_{2,\infty}^{1-\frac{1}{q}}(\mathbb{R}^3) \hookrightarrow H^{\tilde{\rho}}(\mathbb{R}^3)$ with $q > 2$. Thus, the standard Picard's method [10, 13] ensures that the solution (u, b) can be extended after $t = T_0$ in the class $\mathcal{X}(0, T_0)$. This completes the proof of Theorem 1.1. \square

Acknowledgements The authors thank the referees and the associated editor for their invaluable comments and suggestions which helped improve the paper greatly. Q. Chen, C. Miao and Z. Zhang were supported by the NSF of China under grant No.10701012, No.10725102 and No.10601002.

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